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Manuel O Cáceres†

Centro Atómico Bariloche and Instituto Balseiro, CNEA, and Universidad Nacional de Cuyo, Av. Ezequiel Bustillo Km 9.5, 8400 San Carlos de Bariloche, Río Negro, Argentina

E-mail: caceres@cab.cnea.gov.ar

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Abstract. This paper is concerned with a Lévy noise characterized, for $t \in [0, \infty]$, by the functional $G_\xi([k(t)]) = \exp(-b\int_0^\infty |k(s)|^\alpha \, \mathrm{d}s)$, with $0 < \alpha \leqslant 2$. Then Lévy flights can be defined through a stochastic differential equations rather than the usual Chapmann–Kolmogorov equation. We have used this functional approach to solve a plane rotor in the presence of Lévy noise. The linear damped stochastic process driven by Lévy noise is revisited and its non-autonomous and non-Markovian generalizations have been solved in the context of our functional analysis.

1. Lévy noise, Lévy flights

Consider the stochastic process (SP) $\xi(t)$ to be a symmetric (singular) *white noise*, characterized, for $t \in [0, \infty]$, by the functional

$$G_\xi([k(t)]) \equiv \left\langle \exp \mathrm{i} \int_0^\infty \xi(t) k(t) \, \mathrm{d}t \right\rangle = \exp\left(-b \int_0^\infty |k(s)|^\alpha \, \mathrm{d}s\right) \qquad 0 < \alpha \leqslant 2 \quad b > 0. \tag{1.1}$$

The notation $G_\xi([k(t)])$ emphasizes that G depends on the whole test function $k(t)$, and not just on the value it takes at one particular time t_j. The convergence of the integral is accomplished because the real functions $k(t)$ may be restricted to those that vanish for sufficiently large t. For $\alpha = 2$ this functional gives the well known (singular) *Gaussian white-noise* [1].

Proposition. *Non-autonomous Lévy flights [2] can alternatively be interpreted as the SP* $\mathbf{X}(t)$ *defined from the stochastic differential equation (SDE)*:

$$\frac{\mathrm{d}}{\mathrm{d}t} X(t) = C_1(t) + \gamma_2(t)\xi(t) \qquad X \in (-\infty, \infty) \tag{1.2}$$

when the noise $\xi(t)$ *is characterized by the functional (1.1) and* $C_1(t)$, $\gamma_2(t)$ *are sure (non-random) functions of time.*

Proof. Using [3, proposition 3] the functional of the SP $\mathbf{X}(t)$ (1.2), with $t \in [0, \infty]$, can be written in the general form

$$G_X([Z(t)]) = \exp\left(+\mathrm{i}k_0 X_0 + \mathrm{i}\int_0^\infty C_1(t) \int_t^\infty Z(s) \, \mathrm{d}s \, \mathrm{d}t\right) \times G_\xi\left(\left[\gamma_2(s) \int_s^\infty Z(s') \, \mathrm{d}s'\right]\right) \tag{1.3}$$

† Senior Independent Research Associate at CONICET.

where $k_0 = \int_0^\infty Z(t)\,\mathrm{d}t$ and $X_0 \equiv \mathbf{X}(0)$ is the sure initial condition. Then using the functional (1.1) we get

$$G_X([Z(t)]) = \exp\left(+\mathrm{i}k_0 X_0 + \mathrm{i}\int_0^\infty C_1(t)\int_t^\infty Z(s)\,\mathrm{d}s\,\mathrm{d}t\right)$$
$$\times \exp\left(-b\int_0^\infty \left|\gamma_2(s)\int_s^\infty Z(s')\,\mathrm{d}s'\right|^\alpha \mathrm{d}s\right). \tag{1.4}$$

The functional $G_X([Z(t)])$ completely characterizes the SP $\mathbf{X}(t)$. QED

Note that for the autonomous case: $C_1(t) = 0, \gamma_2(t) = 1$, and introducing the test function $Z(t) = k_1\delta(t - t_1)$ in (1.4), the one-time characteristic function of the SP $\mathbf{X}(t)$, with $X_0 = 0$, gives

$$G_X(k_1, t_1) = \exp\left(-b|k_1|^\alpha t_1\right) \tag{1.5}$$

which is just the characteristic function presented by Cauchy [4] in 1853, and later on investigated in detail—regarding its non-negativity—by Lévy [2] in 1925 (for excellent reviews see [5, 6]). From expression (1.5) the stable one-time (conditional) probability distribution $P(x_1, t_1)$ follows by quadrature using Fourier inversion. Closed-form expressions exist for a few values of α other than $\alpha = 2$ (Wiener), $\alpha = 1$ (Cauchy), and the cases $\alpha = \frac{1}{2}$ and $\alpha = \frac{2}{3}$ (Zolotarev [7]) were revisited in the review by Montroll and Bendler [8]. There, several important series expansions have been remarked, in particular and for large x, the power-law distribution $P(x, t) \sim (bt)/x^{1+\alpha}$ can be shown to be valid.

In the present paper we are more interested in the advantages of the closed expression of the functional $G_X([Z(t)])$, and in the statistical self-affine properties of the characteristic function $G_X(k, t)$, rather than in the asymptotic behaviour, for large x, of its one-time distribution $P(x, t)$.

1.1. Non-Markovian generalization

From (1.1) and (1.2) we easily see that the SP $\mathbf{X}(t)$ (Lévy flights) is Markovian because the noise $\xi(t)$ is white. As a matter of fact, this Markovian property is the starting point in the definition of the Lévy flights from the Chapman–Kolmogorov equation, i.e. Lévy flights give rise to the most general Markovian and translational invariant conditional probability distributions [5]. Our alternative *basic* definition stated in (1.1) and (1.2) allow us to generalize Lévy flights $\mathbf{X}(t)$, also, in a non-Markovian framework.

First define the Lévy *correlated noise* by the functional

$$G_\xi([k(t)]) = \exp\left(-\int_0^\infty\int_0^\infty |k(s_1)|^{\alpha/2}|k(s_2)|^{\alpha/2}\,\langle\langle\xi(s_1)\xi(s_2)\rangle\rangle\,\mathrm{d}s_1\,\mathrm{d}s_2\right)$$
$$0 < \alpha \leqslant 2 \tag{1.6}$$

where $\langle\langle\xi(s_1)\xi(s_2)\rangle\rangle$ is in general any suitable function. Hence, once again, using (1.3) and the noise $\xi(t)$ characterized by (1.6), the generalized (non-Markovian) Lévy flights $\mathbf{X}(t)$ are characterized by the functional

$$G_X([Z(t)]) = \exp\left(+\mathrm{i}k_0 X_0 + \mathrm{i}\int_0^\infty C_1(t)\int_t^\infty Z(s)\,\mathrm{d}s\,\mathrm{d}t\right)$$
$$\times \exp\left(-\int_0^\infty\int_0^\infty \left|\gamma_2(s_1)\int_{s_1}^\infty Z(s')\,\mathrm{d}s'\right|^{\alpha/2}\right.$$
$$\left.\times \left|\gamma_2(s_2)\int_{s_2}^\infty Z(s'')\,\mathrm{d}s''\right|^{\alpha/2}\langle\langle\xi(s_1)\xi(s_2)\rangle\rangle\,\mathrm{d}s_1\,\mathrm{d}s_2\right). \tag{1.7}$$

From this functional the whole Kolmogorov hierarchy can—in principle—be calculated by introducing the n-dimensional Fourier transform

$$P(x_1, t_1; x_2, t_2; \ldots; x_n, t_n) = \frac{1}{(2\pi)^n} \int \cdots \int \mathrm{d}k_1 \cdots \mathrm{d}k_n \exp\left(-\mathrm{i} \sum_{i=1}^{n} k_i \; x_i\right) \times \left[G_X\left([Z(t)]\right)\right]_{Z(t)=k_1\delta(t-t_1)+\cdots+k_n\delta(t-t_n)} . \tag{1.8}$$

We remark that (1.7) is an *exact* result which allows us to get the complete characterization of the generalized non-Markovian and non-autonomous Lévy flights $\mathbf{X}(t)$. For example, the one-time conditional probability distribution $P(x_1, t_1)$ is given in terms of $G_X(k_1, t_1)$, which is just the functional $G_X([Z(t)])$ evaluated with the test function $Z(t) = k_1\delta(t - t_1)$.

1.1.1. Autonomous case. The simplest autonomous case corresponds to $C_1(t) = 0$, $\gamma_2(t) = 1$; then from (1.7) and (1.8) we see that to know the one-time probability distribution we first have to calculate the integral

$$\sigma(t_1) \equiv \int_0^{t_1} \int_0^{t_1} \langle\langle \xi(s_1)\xi(s_2)\rangle\rangle \; \mathrm{d}s_1 \, \mathrm{d}s_2. \tag{1.9}$$

Thus the conditional distribution follows by quadrature:

$$P(x_1, t_1|X_0, 0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{d}k_1 \; \exp\left[\mathrm{i}k_1(X_0 - x_1)\right] \exp\left(-|k_1|^{\alpha} \sigma(t_1)\right). \tag{1.10}$$

This Fourier transformation, as before, can only be performed for some particular values of α. As is well known, the notable point concerning the Lévy flights (Markovian or not) is that all the moments of SP $\mathbf{X}(t)$ diverge. This fact can easily be realized from the lack of analyticity (for $\alpha \in (0, 2)$) about the origin, $k_1 = 0$, of the one-time characteristic function (with $X_0 = 0$)

$$G_X(k_1, t_1) = \exp\left(-|k_1|^{\alpha} \sigma(t_1)\right). \tag{1.11}$$

If Lévy flights are non-Markovian the statistical self-affine properties of $G_X(k_1, t_1)$ are lost during the transient regime when no scaling is allowed. Only in the long-time regime and if the correlation $\langle\langle \xi(s_1)\xi(s_2)\rangle\rangle$ is of the short-range class, the function $\sigma(t)$ fulfills the scaling $\sigma(\Lambda t) \to \Lambda\sigma(t)$; then in this case the non-Markovian Lévy flights re-acquire, in the long-time regime, its usual scaling (A.1). A different situation occurs when the correlation $\langle\langle \xi(s_1)\xi(s_2)\rangle\rangle$ is of the long-range class. For example let us assume that the correlation is characterized by the power-law function†

$$\langle\langle \xi(t_1)\xi(t_2)\rangle\rangle = \frac{\Gamma_2 \tau^{-1}}{(1 + |t_1 - t_2|/\tau)^{\mu}} \qquad \tau \in [0, \infty) \quad \mu \geqslant 0. \tag{1.12}$$

The case $\mu = 0$ corresponds to a ballistic-like situation. Then from (1.9) and (1.12) the $\sigma(t)$ function is given, for $0 \leqslant \mu \neq \{1, 2\}$, by

$$\frac{1}{\Gamma_2}\sigma(t) = \frac{2t}{(\mu - 1)} + \frac{2\tau^{\mu-1}(t + \tau)^{2-\mu}}{(\mu - 1)(\mu - 2)} - \frac{2\tau}{(\mu - 1)(\mu - 2)}. \tag{1.13}$$

For $\mu = 1$ we get

$$\frac{1}{\Gamma_2}\sigma(t)\Big|_{\mu=1} = 2\tau + 2(t + \tau)\left(\log\left(1 + \frac{t}{\tau}\right) - 1\right) \tag{1.14}$$

† A similar stationary long-range correlation was also used to study the diffusion-advection equation with a random velocity field [9].

and for $\mu = 2$ reads

$$\frac{1}{\Gamma_2}\sigma(t)\big|_{\mu=2} = 2t - 2\tau \log\left(1 + \frac{t}{\tau}\right). \tag{1.15}$$

Hence, if $0 \leqslant \mu \leqslant 1$ it is simple to see that due to the long-range effect of the correlated Lévy noise (1.6), the one-time characteristic function $G_X(k,t) \equiv \langle \exp \mathrm{i}k\mathbf{X}(t)\rangle$ does not have, even at long times, the familiar Lévy's scaling (A.1). If the noise power-law parameter μ fulfills $\mu \in [0,1)$, we see from (1.13) that a new asymptotic scaling is obtained: $\sigma(\Lambda t) \to \Lambda^{2-\mu}\sigma(t)$. Then from (1.11), the one-time characteristic function fulfills the *asymptotic* long-time scaling

$$G_X\left(\frac{k}{\Lambda^{(2-\mu)/\alpha}}, \Lambda t\right) \longrightarrow G_X(k,t) \qquad \mu \in [0,1) \quad \alpha \in (0,2] \quad t \gg \tau \tag{1.16}$$

which implies, for the non-Markovian Lévy flights $\mathbf{X}(t)$, the asymptotic scaling

$$\mathbf{X}(\Lambda t) \to \Lambda^{(2-\mu)/\alpha}\mathbf{X}(t) \qquad \mu \in [0,1) \quad \alpha \in (0,2] \quad t \gg \tau. \tag{1.17}$$

Then we can conclude that a long-range correlated Lévy noise induces a *strong* non-Markovian effect which changes the long-time *asymptotic scaling* of the SP $\mathbf{X}(t)$. If $\mu = 1$ there are logarithmic corrections. Only if $\mu > 1$ the familiar Lévy scaling (A.1) is re-obtained (in the asymptotic long-time regime).

On the other hand, (1.17) can be used to realize that it is possible to mimic the fractal dimensions of the (usual) Lévy flights by using a strong non-Markovian Gaussian SP characterized by a power-law correlation function as in (1.12), see appendix A.3. Then it follows that for $t \gg \tau$ and $\mu \in [0,1)$ the non-Markovian Gaussian SP $\mathbf{Y}(t)$ (A.2) shares some fractal properties with the persistent Lévy flights, for $\alpha \in [1,2)$, but avoids having divergent moments. Namely, the Gaussian SP $\mathbf{Y}(t)$ has the same fractal dimensions D_B and D as the Lévy flights if we perform assignation (A.6). For the case $\alpha \in (0,1)$ we have not been able to find a completely characterized non-Markovian SP that has the same fractal dimensions as Lévy flights, but avoids having divergent moments.

2. Lévy fluctuations

2.1. Linear dissipative Lévy processes

Our *basic* definition, stated in (1.1) and (1.2), for the Lévy flights $\mathbf{X}(t)$, allows us to solve the interesting problem of a linear system driven by a Lévy noise. In general these types of models can be used when a linear dissipative system is not *efficient* to dissipate the energy supplied by the external noise [10].

Consider the most general non-autonomous linear SDE, characterizing the Markov SP $\mathbf{V}(t)$, of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}V(t) = -\gamma_1(t)V(t) + \gamma_2(t)\xi(t) \qquad V \in (-\infty,\infty) \tag{2.1}$$

where $\gamma_1(t) > 0$ and $\gamma_2(t)$ are sure functions of time, and the random force $\xi(t)$ be characterized by the Lévy (white) noise (1.1). Using [3, proposition 2], the functional of the non-autonomous SP $\mathbf{V}(t)$ is given, for $t \in [0,\infty]$, by

$$\begin{aligned} G_V\left([Z(t)]\right) &= \mathrm{e}^{+\mathrm{i}q_0V_0} G_\xi\left(\left[\gamma_2(t)\int_t^\infty Z(t')\exp\left(\int_{t'}^t \gamma_1(s')\,\mathrm{d}s'\right)\mathrm{d}t'\right]\right) \\ &= \mathrm{e}^{+\mathrm{i}q_0V_0}\exp\left(-b\int_0^\infty \left|\gamma_2(s)\int_s^\infty Z(t')\exp\left(\int_{t'}^s \gamma_1(s')\,\mathrm{d}s'\right)\mathrm{d}t'\right|^\alpha \mathrm{d}s\right) \end{aligned} \tag{2.2}$$

where $q_0 = \int_0^\infty Z(s) \exp\left(-\int_0^s \gamma_1(t)\,\mathrm{d}t\right) \mathrm{d}s$ and $V_0 \equiv \mathbf{V}(0)$ is the (sure) initial condition. Equation (2.2) is an exact expression which—in principle—allows us to completely characterize the SP $\mathbf{V}(t)$. Note that its non-Markovian generalization can also be completely characterized†.

2.1.1. Autonomous case. In the particular autonomous case: $\gamma_1(t) = \gamma$ and $\gamma_2(t) = 1$ the functional $G_V([Z(t)])$ adopts a much more simpler form which is very useful to calculate, in a direct way, the 1− time characteristic function of the SP $\mathbf{V}(t)$, i.e. putting the test function $Z(t) = k_1\delta(t - t_1)$ in (2.2) we obtain

$$G_V\,(k_1, t_1) = \exp\left(b\,\frac{\mathrm{e}^{-\gamma\alpha t_1} - 1}{\gamma\alpha}|k_1|^\alpha + \mathrm{i}k_1\mathrm{e}^{-\gamma t_1}V_0\right). \tag{2.3}$$

The n-time characteristic function $G_V\,(k_1, t_1; k_2, t_2; \cdots; k_n, t_n)$ can in principle be calculated in a similar way, showing in this form the simplicity and elegance of our approach. We should emphasize that the 1− time characteristic function (2.3) was first obtained by Doob through a direct integration procedure [11]. Also West *et al* calculated that one-time characteristic function using the Baker–Campbell–Hausdorff [12] operational method. From (2.3) the one-time conditional distribution $P(V, t)$ is calculated as the Fourier transform of a stable characteristic function (but not translational invariant) with respect to the variable $(V - V_0\mathrm{e}^{-\gamma t})$, this fact was first observed by Doob [11]. Unfortunately, this integration can be performed only for some special values of α. For example, the case $\alpha = 2$ gives the Ornstein–Uhlenbeck process, and $\alpha = 1$ the *damped* Cauchy process; for other values of α only a series solution can be given [10]. The stationary probability distribution corresponds to the Fourier transform of the one-time characteristic function $G_V(k, \infty)$, giving rise in this way to the distribution of Lévy (stationary) fluctuations, i.e. a dynamical system which is able to reach a stationary momentless distribution. This phenomenon was interpreted as the inefficiency of the linear system in dissipating the pumping energy coming from the Lévy noise [10].

The present functional approach not only allows us to calculate the one-time distribution in a direct way; the whole Kolmogorov hierarchy (as we presented in (1.8)) can also, in principle, be obtained by quadrature using the n-dimensional Fourier transform in the functional (2.2), for autonomous as well as non-autonomous cases. This is a clear simplification procedure over the Baker-Campbell-Hausdorff operational method.

2.2. The rigid rotator with Lévy noise

A plane Brownian rotation is a useful model to represent a *spherical* molecule [13] when there is only one relevant variable, the angle $\phi(t)$. Then the angular velocity $\Omega = \mathrm{d}\phi(t)/\mathrm{d}t$ is assumed to follow a Brownian motion where the random torque is represented by and additive Gaussian noise and the dissipation by a given coefficient γ, i.e. $\mathrm{d}^2\phi/\mathrm{d}t^2 + \gamma\mathrm{d}\phi/\mathrm{d}t = \xi(t)$. Instead of having numerous weak collisions, many magnetic systems have strong or very strong collisions (random torque); then the Gaussian plane rotator is unable to describe the transient or the long-time cosine relaxation of such a 'spherical molecule'.

Using our functional approach it is very simple to see that the cosine relaxation $\langle\cos(\phi(t))\rangle$ is exponential, at long times, provided that the random torque is represented by a short-range Gaussian noise. Hence it is worth studying, here, the case when the random torque is characterized by a Lévy noise. Now, because a rigid rotator is equivalent to the translation of a particle on a circular track, this situation corresponds to assuming that the random torque

† In order to work out a non-Markovian generalization of the damped Lévy process $\mathbf{V}(t)$, consider the functional of the noise $\xi(t)$ given by (1.6).

can produce short and very long *angle excursions* (without any characteristic length) on the circular track. Then the evolution equation (Langevin dynamics) of a Lévy plane rotator is defined, here, by the SDEs

$$\frac{\mathrm{d}}{\mathrm{d}t}\Omega(t)+\gamma\Omega(t)=\xi(t)\qquad \frac{\mathrm{d}}{\mathrm{d}t}\phi(t)=\Omega(t)\quad \{\Omega,\phi\}\in(-\infty,\infty) \tag{2.4}$$

where $\xi(t)$ is the Lévy noise characterized by the functional (1.1). These SDEs can be solved, for any noise $\xi(t)$, by using our functional approach as follows.

First, the functional $G_\Omega([k(t)])$ of the angular velocity Ω, equivalent to (2.1), is given in terms of $G_\xi([k(t)])$. Hence the functional of the angle ϕ is equivalent to a generalized Wiener process where, now, the noise is characterized by the functional $G_\Omega([k(t)])$. Therefore the general solution of the stochastic angle $\phi(t)$ is given by the functional

$$G_\phi([Z(t)])=\mathrm{e}^{\mathrm{i}k_0\phi_0}G_\Omega\left(\left[\int_t^\infty Z(s)\,\mathrm{d}s\right]\right) \tag{2.5}$$

where $k_0=\int_0^\infty Z(s)\,\mathrm{d}s$ and ϕ_0 is the *angle* initial condition; so using the explicit expression

$$G_\Omega([M(t)])=\mathrm{e}^{+\mathrm{i}q_0\Omega_0}G_\xi\left(\left[\int_t^\infty \mathrm{e}^{\gamma(t-t')}M(t')\,\mathrm{d}t'\right]\right)$$

we get, with $M(t)=\int_t^\infty Z(t'')\,\mathrm{d}t''$, the functional

$$G_\phi([Z(t)])=\mathrm{e}^{\mathrm{i}k_0\phi_0+\mathrm{i}q_0\Omega_0}G_\xi\left(\left[\int_t^\infty \mathrm{e}^{\gamma(t-t')}\int_{t'}^\infty Z(t'')\,\mathrm{d}t''\,\mathrm{d}t'\right]\right) \tag{2.6}$$

where

$$q_0=\int_0^\infty M(s)\mathrm{e}^{-\gamma s}\,\mathrm{d}s=\int_0^\infty \mathrm{e}^{-\gamma s}\int_s^\infty Z(s')\,\mathrm{d}s'\,\mathrm{d}s$$

and $\Omega_0\equiv\dot\phi_0$ is the *angular velocity* initial condition.

Second, if the noise $\xi(t)$ appearing in (2.4) is a Lévy (white) noise, its functional is given by (1.1); then from (2.6) we get (for $0<\alpha\leqslant 2,\ b>0$) the general solution

$$G_\phi([Z(t)])=\mathrm{e}^{\mathrm{i}k_0\phi_0+\mathrm{i}q_0\Omega_0}\exp\left(-b\int_0^\infty\left|\int_s^\infty \mathrm{e}^{\gamma(s-t')}\int_{t'}^\infty Z(t'')\,\mathrm{d}t''\,\mathrm{d}t'\right|^\alpha\mathrm{d}s\right). \tag{2.7}$$

In general the cosine relaxation function is obtained from the cosine functional†; thus

$$\left\langle\cos\int_0^\infty\phi(t)Z(t)\,\mathrm{d}t\right\rangle=\mathcal{R}_e\left[G_\phi([Z(t)])\right]. \tag{2.8}$$

For the plane Lévy rotator, from (2.7), we see that the one-time characteristic function, of the stochastic angle $\phi(t_1)$ is

$$G_\phi(k_1,t_1)=\exp\left(\mathrm{i}k_1\left[\phi_0+(1/\gamma)\left(1-\mathrm{e}^{-\gamma t_1}\right)\dot\phi_0\right]-b|k_1/\gamma|^\alpha\Sigma(t_1)\right).$$

Then $P(\phi,t)$ is a momentless distribution. Nevertheless, the cosine relaxation is finite, as expected, and is given by

$$\begin{aligned}\langle\cos\phi(t_1)\rangle &= \mathcal{R}_e\left[G_\phi(k_1,t_1)\right]\big|_{k_1=1}=\mathcal{R}_e\left[G_\phi([Z(t)])\right]\big|_{Z(t)=\delta(t-t_1)}\\ &=\cos\left[\phi_0+(1/\gamma)\left(1-\mathrm{e}^{-\gamma t_1}\right)\dot\phi_0\right]\exp\left(-b/\gamma^\alpha\ \Sigma(t_1)\right)\end{aligned} \tag{2.9}$$

where the function $\Sigma(t)$ has the expression:

$$\Sigma(t)=\int_0^t\left|\mathrm{e}^{\gamma(s-t)}-1\right|^\alpha\,\mathrm{d}s. \tag{2.10}$$

† Note that the Gaussian (white) noise case is immediately re-obtained from (2.7) by taking $\alpha=2$.

From (2.10) we see that at long-time, $t \gg \gamma^{-1}$, the behaviour is $\Sigma(t) \sim t$. Then even when the shape of the transient regime of the cosine relaxation can be *anomalous long* (depending on Lévy's exponent $\alpha \in (0, 2)$), in the very long-time limit the relaxation is exponential! Note that the present functional approach also allows us to calculate the two-time cosine correlation $\langle \cos[\phi(t_1) - \phi(t_2)] \rangle$ in a direct way.

The long-range correlated Gaussian case can be work out, in a similar way, considering that the functional of the stochastic angle $\phi(t)$ in this case is

$$G_\phi([Z(t)]) = \mathrm{e}^{\mathrm{i}k_0\phi_0+\mathrm{i}q_0\dot{\phi}_0} \exp\left[-\frac{1}{2}\int_0^\infty \int_0^\infty \left(\int_{s_1}^\infty \mathrm{e}^{\gamma(s_1-t')} \int_{t'}^\infty Z(t'')\,\mathrm{d}t''\,\mathrm{d}t'\right) \times \left(\int_{s_2}^\infty \mathrm{e}^{\gamma(s_2-t')} \int_{t'}^\infty Z(t'')\,\mathrm{d}t''\,\mathrm{d}t'\right) \times \langle\langle \xi(s_1)\xi(s_2)\rangle\rangle\,\mathrm{d}s_1\,\mathrm{d}s_2\right]$$

when the power-law noise correlation $\langle\langle \xi(s_1)\xi(s_2)\rangle\rangle$ has the form (1.12). Then, if $\mu \in [0, 1)$ it is possible to see that the cosine relaxation will not be exponential at long-times. In fact for times $t \gg \tau$ the behaviour looks like

$$\langle \cos\phi(t)\rangle \sim \exp\left(-\frac{2\Gamma_2\tau^{(\mu-1)}\gamma^{-2}}{(1-\mu)(2-\mu)}t^{2-\mu} + \mathcal{O}(t) + \mathcal{O}(t^{2-\mu}\exp(-\gamma t))\right)$$

and hence in this long-time regime the cosine relaxation depends on the time scale of the power-law correlation τ rather than on the dissipative parameter γ. The faster relaxation occurs when $\mu \to 0$, i.e. in the ballistic-like limit. If $\mu = 1$ there are logarithmic corrections; only if $\mu > 1$ will the cosine relaxation, at long time, be exponential.

3. Conclusions

Herein we summarize the main results of the paper.

(1) We have analyzed Lévy flights, the SP $\mathbf{X}(t)$, from a different point of view; i.e. first we defined a Lévy noise $\xi(t)$ and then we solved the characteristic functional of the stochastic differential equation (1.2) driven by that Lévy noise. This corresponds of having completely characterized Lévy flights in terms of its functional $G_X([Z(t)])$. This method also allow us to work out non-autonomous and non-Markovian generalizations of Lévy flights.

(2) The response of a linear dissipative system driven by an external Lévy noise $\xi(t)$ has been revisited, i.e. the non-autonomous damped Lévy process (2.1). The functional of this process $G_V([k(t)]) = \langle \exp \mathrm{i}\int_0^\infty V(t)k(t)\,\mathrm{d}t\rangle$ has been given in (2.2) for arbitrary (sure) time-dependent functions $\gamma_1(t) > 0$ and $\gamma_2(t)$. This fairly general method is based upon knowing the characteristic functional of the noise $G_\xi([k(t)])$, which in the present paper has been assumed to be the Lévy (white) noise (1.1). The particular autonomous case: $\gamma_1(t) = \gamma$ and $\gamma_2(t) = 1$ has been explicitly worked out, and its one-time characteristic function shown to be in agreement with previous results [10, 11]. From this one-time characteristic function the conditional probability density, for arbitrary α, follows as series representation for large $(V - \mathrm{e}^{-\gamma t}V_0)$.

(3) The plane rotator (2.4) in presence of a Lévy random torque $\xi(t)$ has been solved. We have showed that the cosine relaxation has a *very* long transient regime, and we have proved that its shape depends on Lévy's parameter α, (but in the long-time limit the relaxation is exponential). Hence in the transient regime, the cosine relaxation can mimic a non-exponential behaviour, as is the case in magnetic systems of nanometer size [16].

On the other hand, the long-range Gaussian case has also been solved, showing in this way that at long-times the cosine relaxation is non exponential: $e^{-Ct^{\beta}}$ with $\beta \in (1, 2]$ and $C =$ constant.

(4) Some statistics self-affine signatures of Lévy flights (for $\alpha \in [1, 2)$) have shown to be equivalent to the ones calculated from a strong non-Markovian Gaussian SP $\mathbf{Y}(t)$ (in the long-time regime $t \gg \tau$). This fact has been pointed out, in the appendix, analyzing the asymptotic scaling and the increments of a persistent non-Markovian stochastic process (NMSP) $\mathbf{Y}(t)$, in order to calculate its fractal dimensions D_B and D. If the long-range noise parameter fulfills $\mu \in [0, 1)$, past increments of the NMSP $\mathbf{Y}(t)$ are correlated with future increments, so a long-range correlated Gaussian noise induces infinitely long-run correlations in the NMSP $\mathbf{Y}(t)$ as in the *persistent* fBm [17].

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Appendix. Fractal dimensions

A.1. Box dimension

Let us go back to the usual Lévy flights (Markovian and autonomous), i.e. those characterized by the functional (1.3) with $C_1(t) = 0$, $\gamma_2(t) = 1$. In this case the SP $\mathbf{X}(t)$ is statistical self-affine for all times. This fact can be seen from the k, t structure of the one-time characteristic function $G_X(k, t)$, see (1.5). From this expression, the following scaling invariance—in distribution—is true $G_X(\Lambda^{-1/\alpha}k, \Lambda t) = G_X(k, t), \ \Lambda > 0$. Hence Lévy flights $\mathbf{X}(t)$ fulfill the *scaling*

$$\mathbf{X}(\Lambda t) = \Lambda^{1/\alpha}\mathbf{X}(t). \tag{A.1}$$

From this scaling rule the *box dimension* D_B can be evaluated 'mechanically' for a set of points such as the record of the SP $\mathbf{X}(t)$. To calculate this fractal dimension we closely follow Feder's arguments [18]. Let the time-span of the record be $\mathcal{T}$, then we need $\mathcal{T}/\Lambda t$ segments of length Λt to cover the time axis. Hence from the scaling (A.1) it follows that

$$\Delta\mathbf{X}(\Lambda t) \equiv \mathbf{X}(\Lambda t) - \mathbf{X}(t_0) = \Lambda^{1/\alpha}\,(\mathbf{X}(t) - \mathbf{X}(t_0)) \equiv \Lambda^{1/\alpha}\,\Delta\mathbf{X}(t)$$

thus in each segment the range of the record is of the order $\Delta\mathbf{X}(\Lambda t) = \Lambda^{1/\alpha}\,\Delta\mathbf{X}(t)$ and we need a stack of $\Lambda^{1/\alpha}\,\Delta\mathbf{X}(t)/\Lambda a$ boxes of height Λa to cover that range. Therefore the number of boxes to cover the set is of the order

$$\mathcal{N}(\Lambda, a, t) = \frac{\Lambda^{1/\alpha}\,\Delta\mathbf{X}(t)}{\Lambda a} \times \frac{\mathcal{T}}{\Lambda t} \sim \Lambda^{(1/\alpha)-2}$$

which leads to the box counter fractal dimension $D_B = 2 - 1/\alpha$, for $\alpha \in (1, 2]$. Note that in this argument we have used boxes that were small with respect to both the length of the record $\mathcal{T}$ and the range of the record, thus this relation holds in high-resolution, so this is a *local* fractal dimension. In the case $\alpha \to 2$ the Lévy SP $\mathbf{X}(t)$ approaches Wiener's box counter fractal dimension $D_B \to \frac{3}{2}$.

A.2. Divider dimension

Another fractal dimension that also can be evaluated 'mechanically' is the *divider dimension* along a curve to measure its length. For self-similar fractal curves such as coastlines this fractal dimension can be estimated from the behaviour of its total length $L \sim \delta^{1-D}$, where δ is the 'length' of the rule. The measured length, of the path in the x, t plane, with a rule of length δ, placed such that it covers a time step Λt gives a contribution to the length

$$\delta = \sqrt{(\Lambda t)^2 + \left(\frac{\Delta \mathbf{X}(\Lambda t)}{a}\right)^2} = \sqrt{(\Lambda t)^2 + \Lambda^{2/\alpha}\left(\frac{\Delta \mathbf{X}(t)}{a}\right)^2}$$

where the last equality was written by virtue of (A.1). Here, as before, a measures the scale on the x axis. Then, depending on the magnification in the x axis the behaviour of δ as a function of Λ will be different. Using a small a, the dominant behaviour is $\delta \sim \Lambda^{1/\alpha}$. Hence the number of segments along the time axis is $\mathcal{T}/\Lambda t \sim \Lambda^{-1} \sim \delta^{-\alpha}$, where the last equality was written by virtue of the small a dominant behaviour. Therefore the total length will have the behaviour $L = (\mathcal{T}/\Lambda t) \times \delta \sim \delta^{1-\alpha}$, from which the *divider fractal dimension* [14, 18] reads $D = \alpha$, for $\alpha \in (0.2]$. This last result could also be found from the power-law asymptotic form associated with the distribution of Lévy flights for large x. Then from this asymptotic form, the cumulative function gives $\mathcal{P}(x > \eta) \sim \eta^{-\alpha}$, and from this hyperbolic random variable the fractal dimension of the trajectory $D = \alpha$ follows [15].

The advantage of calculating D_B and D from the scaling invariance of the one-time characteristic function is because we can use this method for a very different SP So we may wonder which could be the SP $\mathbf{Y}(t)$ that not having divergent moments, has same of the statistical self-affine properties of the Lévy flights $\mathbf{X}(t)$, i.e. for example the fractal dimensions D_B and D. To proceed with this program we will introduce a particular non-Markovian Gaussian SP $\mathbf{Y}(t)$ with long-range correlations.

A.3. A strong non-Markovian Gaussian process

Let the equation of motion of a one-dimensional Brownian-like particle in a generalized medium be

$$\frac{\mathrm{d}Y}{\mathrm{d}t} = \xi(t) \qquad Y \in (-\infty, \infty) \tag{A.2}$$

where $\xi(t) \in \mathrm{Re}$ is a Gaussian long-range correlated *noise*, so its correlation function can be characterized by a power-law function as in (1.12). Hence it follows that the functional of the non-Markovian stochastic process (NMSP) $\mathbf{Y}(t)$ is

$$G_Y([Z(t)]) = \mathrm{e}^{+\mathrm{i}k_0 Y_0} \exp\left[-\frac{1}{2}\int_0^\infty \int_0^\infty \left(\int_{s_1}^\infty Z(s)\,\mathrm{d}s\right)\left(\int_{s_2}^\infty Z(s')\,\mathrm{d}s'\right) \times \langle\langle \xi(s_1)\xi(s_2)\rangle\rangle \,\mathrm{d}s_1\,\mathrm{d}s_2\right] \tag{A.3}$$

where $Y_0 \equiv \mathbf{Y}(0)$ is the (sure) initial condition, and $k_0 = \int_0^\infty Z(s)\,\mathrm{d}s$. From (A.3) using the test function $Z(t) = k_1\delta(t - t_1)$, the one-time characteristic function reads $G_Y(k_1, t_1) = \exp(-k_1^2\sigma(t_1) + \mathrm{i}k_1Y_0)$, where the dispersion $\sigma(t)$ is given, for $0 \leqslant \mu \neq 1, 2$, in (1.13). If $0 \leqslant \mu \leqslant 1$ it is possible to see that due to the long-range effect of the Gaussian noise $\xi(t)$, the one-time characteristic function $G_Y(k, t) \equiv \langle\exp(\mathrm{i}k\mathbf{Y}(t))\rangle$ does not, even at long times, have Wiener's scaling. If the power-law parameter μ fulfills $\mu \in [0, 1)$, we see from (1.13) that a new asymptotic scaling is obtained: $\sigma(\Lambda t) \to \Lambda^{2-\mu}\sigma(t)$. Then the one-time characteristic function of NMSP $\mathbf{Y}(t)$ fulfills the *asymptotic* long-time scaling (with $\mathbf{Y}_0 = 0$)

$$G_Y\left(\frac{k}{\sqrt{\Lambda^{2-\mu}}}, \Lambda t\right) \longrightarrow G_Y(k, t) \qquad \mu \in [0, 1) \quad t \gg \tau \tag{A.4}$$

which implies in the NMSP $\mathbf{Y}(t)$ a super-diffusion asymptotic scaling:

$$\mathbf{Y}(\Lambda t) \to \sqrt{\Lambda^{2-\mu}}\mathbf{Y}(t) \qquad \mu \in [0, 1) \quad t \gg \tau. \tag{A.5}$$

From the *asymptotic* scaling (A.5), and the scaling of Lévy flights (A.1) it follows that if we perform the assignation†

$$\frac{1}{\alpha} = 1 - \frac{\mu}{2} \qquad \text{for } \alpha \in [1, 2), \ \mu \in [0, 1) \tag{A.6}$$

both processes will have the same fractal dimensions D_B and D. Hence we have found a completely characterized NMSP $\mathbf{Y}(t)$ which shares some statistical self-affine properties with Lévy flights, for the case $\alpha \in [1, 2)$, but which avoids having divergent moments.

Alternatively, the persistence of the NMSP $\mathbf{Y}(t)$ can be interpreted from (A.3). It is straightforward to calculate the correlation function of future increments $[\mathbf{Y}(t) - \mathbf{Y}(0)]$ with past increments $[\mathbf{Y}(0) - \mathbf{Y}(-t)]$ (for $0 \leqslant \mu \neq \{1, 2\}$):

$$\begin{aligned} C(t) &\equiv \frac{1}{\langle \mathbf{Y}(t)^2 \rangle} \langle [\mathbf{Y}(0) - \mathbf{Y}(-t)][\mathbf{Y}(t) - \mathbf{Y}(0)] \rangle \\ &= \frac{\tau + \tau^{\mu-1}(\tau + 2t)^{2-\mu}}{2(\mu - 2)t - 2\tau + 2\tau^{\mu-1}(\tau + t)^{2-\mu}}. \end{aligned} \tag{A.7}$$

Therefore if $0 \leqslant \mu < 1$, past increments are correlated with future increments, i.e. a long-range correlated noise with $\mu \in [0, 1)$ induces infinitely long-run correlations in the NMSP $\mathbf{Y}(t)$ like in the persistent fBm (super-diffusion [17]). In contrast, if $\mu > 1$ the normalized correlation function $C(t)$ goes to zero in the limit $t \to \infty$, in agreement with a Wiener-like behaviour. It is also possible to see, using the fact the noise $\xi(t)$ is symmetric and adopting the initial condition $\mathbf{Y}(0) = 0$, that the variance of an arbitrary increment of the NMSP $\mathbf{Y}(t)$ is given (for $0 \leqslant \mu \neq \{1, 2\}$ and assuming $t_1 \leqslant t_2$) by

$$\left\langle [\mathbf{Y}(t_2) - \mathbf{Y}(t_1)]^2 \right\rangle = \frac{2\Gamma_2}{(\mu - 1)(\mu - 2)} \left[(t_2 - t_1)(\mu - 2) - \tau + \tau^{\mu-1}(\tau + t_2 - t_1)^{2-\mu} \right]. \tag{A.8}$$

Thus, for $t_2 - t_1 \gg \tau$ and if the noise parameter $\mu \in [0, 1)$, we see that $\langle [\mathbf{Y}(t_2) - \mathbf{Y}(t_1)]^2 \rangle$ increases with time as $\sim (t_2 - t_1)^{2-\mu}$ in agreement with the picture of a persistent fBm. If the noise parameter is $\mu > 1$ we re-obtain—in the asymptotic long-time regime—Wiener's result $\langle [\mathbf{Y}(t_2) - \mathbf{Y}(t_1)]^2 \rangle \sim |t_2 - t_1|$.

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† It should be pointed out that in Feder's argument Λ is supposed to be a small parameter. So in order to apply this argument to our NMSP $\mathbf{Y}(t)$ we must assume that we have allowed the process to run for a long time so that any transient has disappeared, $t \gg \tau$. Therefore the portion of the record (time-span of the record $\mathcal{T}$) that we want to measure is in fact in the long-time regime, so it fulfills the *asymptotic* scaling.

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